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Strong Independent Saturation in Complementary Prisms

Zeynep Nihan Berberler

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Abstract
The strong independent saturation number $I^s(G)$ of a graph $G = (V, E)$ is defined as $\min\{I^s(v) : v \in V\}$, where $I^s(v)$ is the maximum cardinality of a minimal strong independent dominating set of G that contains v. Let $\bar{G}$ be the complement of a graph G . The complementary prism $\bar{G}G$ of G is the graph formed from the disjoint union of G and $\bar{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\bar{G}$. In this paper, the strong independent saturation in complementary prisms are considered, complementary prisms with small strong independent saturation numbers are characterized, and relationship between strong independent number and the distance-based parameters are investigated.

Keywords: *Complementary Prisms; Independence; Strong Independent Saturation*

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1. Introduction

Graph theoretic techniques provide a convenient tool for the investigation of networks. It is well-known that an interconnection network can be modeled by a graph with vertices representing sites of the network and edges representing links between sites of the network. Therefore various problems in networks can be studied by graph theoretical methods. Independence based parameters reveal an underlying efficient and stable communication network. A subset of pairwise nonadjacent vertices in a graph G is called independent (or stable/internally stable). The cardinality of a maximum size independent set in G is called the independence (or stability) number (or coefficient of internal stability [7]) of G and is denoted by $\beta(G)$. The independence number of a graph is one of the basic numerical characteristics of a graph and most fundamental and well-studied graph parameters. The maximum stable set problem is a central problem in combinatorial optimization, and has been the subject of extensive study. The problem of determining a stable set of maximum cardinality is a basic algorithmic graph problem occurring in many models in computer science and operations research and finds important applications in various fields, including computer vision and pattern recognition. Finding a maximum independent set is a well-known widely-studied *NP* -hard problem. We refer to [2] for a review concerning algorithms, applications, and complexity issues of this problem.

Among the independence-type parameters that have been studied, the strong independent saturation number is one of the fundamental ones introduced in [8]. In a graph $G(V(G), E(G))$, a subset $S \subseteq V$ of vertices is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex of S . Let $u, v \in V$. Then, u strongly dominates v and if $uv \in E$ and $deg(u) \geq deg(v)$. A set $D \subset V$ is a strong-dominating set of G if every vertex in $V - D$ is strongly dominated by at least one vertex in D [9]. A strong independent dominating set is both independent and strong-dominating. For a vertex v of a graph G , $I^s(v)$ denotes the maximum cardinality of a

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minimal strong independent dominating set of G which contains v . The strong independent saturation number of G, denoted by $I^s(G)$, is the value $\min\{I^s(v) : v \in V\}$. Let $v \in V$ be such that $I^s(V) = I^s(G)$. Then any minimal strong independent dominating set of cardinality $I^s(G)$ containing v is called an I^s -set.

In this paper, finite undirected graphs without loops and multiple edges are considered. The order of G is the number of vertices in G . The open neighborhood of v is $N(v) = \{u \in V(G)|uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The degree of a vertex v is $deg_G(v) = |N(v)|$. A vertex of degree zero is an isolated vertex or an isolate. A leaf or an endvertex or a pendant vertex is a vertex of degree one and its neighbor is called a support vertex. The maximum degree of G is $\Delta(G) = \max\{deg_G(v)|v \in V(G)\}$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. The eccentricity of a vertex v in G is the distance from v to a vertex farthest away from v in G. The diameter of G , denoted by $diam(G)$, is the largest distance between two vertices in $V(G)$ [3–5, 10].

Complementary prisms were first introduced by Haynes, Henning, Slater and Van der Merwe in [6]. For a graph G , its complementary prism, denoted by $G\bar{G}$, is formed from a copy of G and a copy of $\bar{G}$ by adding a perfect matching between corresponding vertices. For each $v \in V(G)$, let $\bar{v}$ denote the vertex v in the copy of $\bar{G}$. Formally $G\bar{G}$ is formed from $G \cup \bar{G}$ by adding the edge $v\bar{v}$ for every $v \in V(G)$. For example, if G is a 5-cycle, then $G\bar{G}$ is the Petersen graph. Also, independence saturation of complementary prisms are studied in detail in [1].

The corona of a graph G , denoted by $G \circ K_1$, is a graph constructed from a copy of the graph G where each vertex of $V(G)$ is adjacent to exactly one vertex of degree one.

We use $\lfloor x \rfloor$ to denote the largest integer not greater than x , and $\lceil x \rceil$ to denote the least integer not less than x .

The paper proceeds as follows. In section 2, existing literature on strong independent saturation number is reviewed. The strong independent saturation numbers for complementary prism $G\bar{G}$ when G is a specified family of graphs are computed and formulae are derived. Strong independent saturation numbers for the graphs and vertices with specific distance-based parameters are investigated. The graphs and vertices for which I^s is small are characterized.

2. Strong independent saturation

2.1 Known results

Theorem 2.1. *[8] The strong independent saturation of*

(a) *the complete graph* K_n *is* 1;

(b) *the path* P_n *is* $\begin{cases} \lfloor n/2 \rfloor \, , if \; n \geq 2, n \neq 3; \\ 0, \; n = 3. \end{cases}$;

(c) *the cycle* C_n $(n \geq 3)$ *is* $\lfloor n/2 \rfloor$;

(d) *the complete bipartite graph* $K_{m,n}$ *is* $\begin{cases} n \, , if \; m = n; \\ 0, \; otherwise. \end{cases}$;

(e) *the double star* $D_{r,s}$ *is* $\begin{cases} r+1 \, , if \; r = s; \\ 0, \; otherwise. \end{cases}$.

2.2 Complementary prisms

We begin this subsection by determining the strong independent saturation number of the complementary prism $G\bar{G}$ when G is a specified family of graphs.

Observation 2.2.

(a) $\beta(P_n) = \lceil n/2 \rceil$

(b) $\beta(C_n) = \lfloor n/2 \rfloor$

(c) $For\ \ n > 3,\ \beta(\bar{C_n})\ = 2$

Definition 2.1. [2] Let G_1 and G_2 be two disjoint graphs. The union of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$.

Observation 2.3. Let $G_1, G_2, ..., G_n$ be disjoint graphs. If $G = \bigcup_{i=1}^{n} G_i$, then $\beta(G) = \sum_{i=1}^{n} \beta(G_i)$.

Theorem 2.4.

(a) If $G = K_n,$ *then* $I^s(G\bar{G}) = n.$

(b) If $G = tK_2 \quad (t > 1),$ *then* $I^s(G\bar{G}) = t + 1.$

(c) If $G = K_t \circ K_1,$ *then* $I^s(G\bar{G}) = t + 1.$

(d) If $G = K_{1,n} \quad (n > 1),$ *then* $I^s(G\bar{G}) = 0.$

(e) If $G = K_{m,n}$ *where* $2 \leq m \leq n,$ *then* $I^s(G\bar{G}) = \begin{cases} m+1\ , if\ m = n; \\ 0,\ if\ m < n. \end{cases}$.

(f) If $G = C_n \quad (n > 3),$ *then* $I^s(G\bar{G}) = \lceil (n+2)/2 \rceil.$

(g) If $G = W_n \quad (n > 3),$ *then* $I^s(G\bar{G}) = 0.$

(h) If $G = P_n \quad (n > 3),$ *then* $I^s(G\bar{G}) = 0.$

Proof. To prove (a), for $G = K_n$, the complementary prism $G\bar{G}$ is the corona $K_n \circ K_1$. Let v be a vertex of $G\bar{G}$. If v is a support vertex, then the $I^s(v)$ -set includes each leaf of the other support vertices except v . If v is a leaf, then $I^s(v)$ -set has a support vertex which is not adjacent to v and each leaf of the other support vertices. Hence, for any $v \in V(G\bar{G})$, $I^s(v) = n$ and thus $I^s(G\bar{G}) = n$.

To prove (b), label the $2t$ vertices of $V(G)$ as u_i, v_i where $1 \leq i \leq t$ such that $u_iv_i \in E(G)$. Let I be a $I^s(v)$ -set of $G\bar{G}$. Then, there exist three cases depending on the different type of vertices of $G\bar{G}$:

Case 1. If $v = \bar{u_i}$ or $v = \bar{v_i}$, then $I = \{\bar{u_i}\} \cup \{\bar{v_i}\} \cup \{u_j\ :\ \forall j \neq i\}$ with cardinality $t + 1$.

Case 2. If $v = u_i$ then being $t \neq i$, then $I = \{\bar{u_t}\} \cup \{\bar{v_t}\} \cup \{u_i\ :\ \forall i \neq t\}$ with cardinality $t + 1$.

Case 3. If $v = v_i$ then being $t \neq i$, then $I = \{\bar{u_t}\} \cup \{\bar{v_t}\} \cup \{v_i\ :\ \forall i \neq t\}$ with cardinality $t + 1$.

By Cases 1, 2, and 3, for any $v \in V(G\bar{G})$, $I^s(v) = t + 1$, and we have $I^s(G\bar{G}) = t + 1$.

To prove (c), let $G = K_t \circ K_1$. If $t = 1$, then $G = K_2$ and from (a), $I^s(G\bar{G}) = 2$. Assume that $t \geq 2$, and label the vertices of G as follows :let $A = \{a_i \mid 1 \leq i \leq t\}$ be the set of t vertices that induce the subgraph K_t of G, and let $B = \{b_i \mid 1 \leq i \leq t\}$ be the end-vertices in G adjacent to the vertices in A such that $a_ib_i \in E(G)$. Then, there exist three cases depending on the different type of vertices of $G\bar{G}$:

Case 1. If $v = \bar{b_i}$ or $v = a_i$, then $I^s(v)$-set is composed of $\{\bar{b_i}\} \cup \{a_i\} \cup B \setminus \{b_i\}$ with cardinality $t + 1$.

Case 2. If $v = \bar{a_i}$, then being $j \neq i$, $I^s(v)$ -set is composed of $\{\bar{a_i}\} \cup \{\bar{b_i}\} \cup \{a_j\} \cup B \setminus \{b_i, b_j\}$ with cardinality $t + 1$.

Case 3. If $v = b_i$, then being $j \neq i$, $I^s(v)$ -set is composed of $\{b_i\} \cup \{\bar{b_j}\} \cup \{a_j\} \cup B \setminus \{b_i, b_j\}$ with cardinality $t + 1$.

By Cases 1, 2, and 3, since for all $v \in V(G\bar{G})$, $I^s(v) = t + 1$, and we have that $I^s(G\bar{G}) = t + 1$.

To prove (d), since G is a star, the support vertex t in G is an isolated vertex $\bar{t}$ in $\bar{G}$ and a leaf in $G\bar{G}$. Denote the n leaves of G by $\{u_i \mid 1 \leq i \leq n\}$. The leaves in G will form a complete graph on n vertices in $\bar{G}$. Let I be a $I^s(v)$ -set of $G\bar{G}$. Then, there exist four cases according to the types of vertices in $G\bar{G}$.

Case 1. If $v = t$, then $I = \{t, \bar{u_i}\}\ (1 \leq i \leq n)$ with $I^s(v) = 2$.

Case 2. If $v = \bar{t}$, then we have that $N_{G\bar{G}}(v) = \{t\}$. However, vertex t cannot be strong dominated by any other vertex either vertex v or vertex u_i $(\forall i)$ where $\{u_i : \forall i\} \in N_{G\bar{G}}(t)$. Therefore, there is no minimal independent strong dominating set containing v yielding $I^s(v) = 0$.

Case 3. If $v = u_i \quad (1 \le i \le n)$, then we have that $N_{G\bar{G}}(v) = \{t, \bar{u_i}\}$. Therefore, similar to Case 2, $I^s(v) = 0$.

Case 4. If $v = \bar{u_i} \quad (1 \le i \le n)$, then $I = \{t, u_i\}$ is the unique minimal independent strong dominating set containing v, so $I^s(v) = 2$.

Consequently, by Cases 1, 2, 3, and 4, $\quad I^s(G\bar{G}) = min\{2, 0\} = 0$.

To prove (e), let $G = K_{m,n}\,(2 \le m \le n)$, where R and S are the partite sets of G with cardinality m and n , respectively. Let $R = \{r_1, r_2, ..., r_m\}$ and $S = \{s_1, s_2, ..., s_n\}$. The vertices of R and S will form complete graphs K_m and K_n on m and n vertices, respectively, in $\bar{G}$. Then, there exist four cases depending on the types of vertices of $G\bar{G}$:

Case 1. If $r_i \in R \quad (1 \le i \le m)$, then $I = \{\bar{s}\} \cup R$ is the $I^s(r_i)$ -set yielding $I^s(r_i) = m + 1$.

Case 2. If $s_i \in S \quad (1 \le i \le n)$, then two subcases occur:

Subcase 1. If $m = n$, then $I = S \cup \{\bar{r}\}$ is the $I^s(s_i)$-set with cardinality $n + 1$.

Subcase 2. If $m < n$, then we have that $N_{G\bar{G}}(s_i) = R \cup \{\bar{s_i}\}$. Since for any vertex $r \in R$, $deg_{G\bar{G}}(r) = n + 1$, that is, $deg_{G\bar{G}}(s_i) = m + 1 < deg_{G\bar{G}}(r)$, vertex r cannot be strong dominated by vertex s_i . In addition, being $N_{G\bar{G}}(r) = S \cup \{\bar{r}\}$ and $deg_{G\bar{G}}(\bar{r}) = m < deg_{G\bar{G}}(r)$, vertex r cannot be strong dominated by $\bar{r}$, either. Hence, there does not exist a minimal independent strong dominating set containing the vertex s_i , yielding $I^s(s_i) = 0$.

Case 3. If $\bar{r_i} \in \bar{R}\ (1 \le i \le m)$, then two subcases occur:

Subcase 1. If $m = n$, then $I = S \cup \{\bar{r_i}\}$ is the unique $I^s(\bar{r_i})$ -set with cardinality $n + 1$.

Subcase 2. If $m < n$, then we have that $N_{G\bar{G}}(\bar{r_i}) = r_i \cup \bar{R} \backslash \{\bar{r_i}\}$. On one hand, since $deg_{G\bar{G}}(r_i) = n+1 > deg_{G\bar{G}}(\bar{r_i}) = m$, vertex r_i cannot be strong dominated by vertex $\bar{r_i}$. On the other hand, $N_{G\bar{G}}(r_i) = \{\bar{r_i}\} \cup S$ but since $deg_{G\bar{G}}(r_i) > deg_{G\bar{G}}(s) = m + 1$, vertex r_i cannot be strong dominated by any vertex s , either, implying that $I^s(\bar{r_i})$-set is an empty set, thus $I^s(\bar{r_i}) = 0$.

Case 4. If $\bar{s_i} \in \bar{S}\ (1 \le i \le n)$, then $I = R \cup \{\bar{s_i}\}$ is the unique $I^s(\bar{s_i})$ -set with cardinality $m + 1$.

As a consequence, by Cases 1, 2, 3, and 4,
if $m = n$, then $\quad I^s(G\bar{G}) = min\{m + 1, n + 1\} = m + 1 = n + 1$;
if $m < n$, then $\quad I^s(G\bar{G}) = min\{m + 1, 0\} = 0$. Thus, the proof holds.

To prove (f), let the vertices of $G = C_n$ be labeled sequentially as $u_0, u_1, ..., u_{n-1}$. There exist two cases according to the number of vertices of C_n :

Case 1. If n is even;

Subcase 1. For $n \neq 4$;
If I is a $I^s(u_j)$-set of $G\bar{G}$ where $0 \le j \le n - 1$, then exactly two vertices of $\bar{G}$ different than the vertex u_j should be in I to independent strong dominate $\bar{V}$. Let the two vertices be $\bar{u}_{j+1}$ and $\bar{u}_{j+2}$, where $j + 1$ and $j + 2$ are taken modulo n , and $S = V(G\bar{G}) \setminus \{N_{G\bar{G}}[\bar{u}_j] \cup N_{G\bar{G}}[\bar{u}_{j+1}] \cup N_{G\bar{G}}[\bar{u}_{j+2}]\}$. Since $G\bar{G}[S] = P_{n-4}$, I includes the independent set of maximum cardinality of $G\bar{G}[S]$, yielding $I^s(u_j) = 3 + \beta(P_{n-4}) = (n + 2)/2$.
If I is a $I^s(\bar{u_j})$ -set of $G\bar{G}$ where $0 \le j \le n-1$, then let $S = V(G\bar{G}) \backslash N_{G\bar{G}}[\bar{u}_j]$. We have that $G\bar{G}[S] = C_{n+1}$. Since the two adjacent vertices $\bar{u}_{j-1}$ and $\bar{u}_{j+1}$ are both of degree $n - 2$ where $j - 1$ and $j + 1$ are taken modulo n , by including one of them to set I , other one and the vertex $\bar{u}$ of degree 3 in $G\bar{G}$ are all strong dominated. Thus, $I^s(\bar{u_j}) = 1 + \beta(C_{n+1}) = (n + 2)/2$.

Subcase 2. For $n = 4$;
If I is a $I^s(u_j)$-set of $G\bar{G}$ where $0 \leq j \leq n-1$, then since $deg_{G\bar{G}}(u) > deg_{G\bar{G}}(\bar{u})$ for all $u \in V$ and $\bar{u} \in \bar{V}$, I exactly includes the vertices u_j and u_{j+2} , where $j+2$ is taken modulo n. Let $S = V(G\bar{G}) \setminus \{N_{G\bar{G}}[u_j] \cup N_{G\bar{G}}[u_{j+2}]\}$. Hence, $G\bar{G}[S] = P_2$ and obviously $I^s(u_j) = 2 + \beta(P_2) = 3$.
If I is a $I^s(\bar{u_j})$-set of $G\bar{G}$ where $0 \leq j \leq n-1$, then let $S = V(G\bar{G}) \setminus N_{G\bar{G}}[\bar{u}_j]$. We have that $G\bar{G}[S] = C_{n+1}$. $N_{G\bar{G}}(\bar{u}_j) = \{u_j, \bar{u}_{j+2}\}$ being $deg_{G\bar{G}}(\bar{u}_j) = 2$ where $j+2$ is taken modulo n , thus vertex u_j of degree 3 cannot be strong dominated by vertex $\bar{u_j}$. Therefore, I also includes both vertices u_{j-1} and u_{j+1} , where $j-1$ and $j+1$ are taken modulo n , in order to be a minimal independent strong dominating set , yielding $I^s(\bar{u_j}) = \{\bar{u}_j, u_{j-1}, u_{j+1}\}$ with cardinality 3.

By $Subcases$ 1 and 2, for n is even, $I^s(G\bar{G}) = (n+2)/2$.

Case 2. If n is odd;
Similar to $Subcase$ 1 of $Case$ 1, $I^s(u_j) = 3 + \beta(P_{n-4}) = (n+3)/2$ and $I^s(\bar{u_j}) = 3 + \beta(C_{n+1}) = (n+3)/2$. Therefore, for n is odd, $I^s(G\bar{G}) = (n+3)/2$.

By $Cases$ 1 and 2, $I^s(G\bar{G}) = \lceil (n+2)/2 \rceil$. Thus, the proof holds.

To prove (g), let G be a wheel of order $n+1$ and consider $G\bar{G}$. Since the center vertex c of G is adjacent to every other vertex of G, it is an isolate in $\bar{G}$ and a leaf in $G\bar{G}$. Therefore, if I is a $I^s(c)$-set of $G\bar{G}$, then first $c \in I$. Since $W_n = C_n + K_1$, if the vertices of C_n are labeled sequentially as $u_0, u_1, ..., u_{n-1}$ in $G\bar{G}$, then $N_{G\bar{G}}(c) = \{\bar{c}\} \cup \{u_i \mid 0 \leq i \leq n-1\}$. Let $S = V(G\bar{G}) \setminus N_{G\bar{G}}[c]$. Eventually, $G\bar{G}[S] = \bar{C}_n$ and so $I^s(c) = 1 + \beta(\bar{C}_n) = 3$.

Now, consider the leaf $\bar{c}$ in $G\bar{G}$. $N_{G\bar{G}}[\bar{c}] = \{c\}$ and $deg_{G\bar{G}}(c) = n+1 > deg_{G\bar{G}}(\bar{c}) = 1$, that is, vertex $\bar{c}$ cannot strong dominate the support vertex c. In addition, $N(c) = \{\bar{c}\} \cup \{u_i \mid \forall i\}$ and since $deg_{G\bar{G}}(u_i) = 4 < deg_{G\bar{G}}(\bar{c})$, it is impossible to strong dominate vertex c by either vertex $\bar{c}$ or any vertex u_i , yielding $I^s(\bar{c}) = 0$.

Consider the vertices of C_n which are labeled sequentially as $u_0, u_1, ..., u_{n-1}$ in $G\bar{G}$. For any vertex $u_i (0 \leq i \leq n-1), c \in N(u_i)$. However, we just investigated that the center vertex c of G cannot be strong dominated in $G\bar{G}$ by any other neighbor vertex, therefore $I^s(u_i) = 0$.

Consequently, consider the vertices of $\bar{C}_n$ in $G\bar{G}$. For a vertex $\bar{u}_i \quad (0 \leq i \leq n-1)$ of $G\bar{G}$, $\bar{u}_i$ strong dominates all its neighbors in $\bar{G}$, those are the vertices in $\bar{G}$ except $\bar{u}_{i-1}$ and $\bar{u}_{i+1}$, where $i-1$ and $i+1$ are taken modulo n . Since center vertex c of G cannot be strong dominated by any of its neighbors, c is included in $I^s(\bar{u}_i)$-set and so vertices $\bar{c}$, c , and $u_i \ (\forall i)$ are all strong dominated. Thus, $I^s(\bar{u}_i) = 1 + \beta(\bar{C}_n)$ and $I^s(\bar{u}_i) = 3$.
As a result, $I^s(G\bar{G}) = min\{3, 0\} = 0$.

To prove (h), we have two cases depending on the number of vertices of P_n :

Case 1. n is odd;
Consider the vertices of $\bar{P}_n$ in $G\bar{G}$.
If a vertex $\bar{v}$ of $\bar{P}_n$ in $G\bar{G}$ is an endvertex of P_n , then $\bar{v}$ all strong dominates its neighbors. Let $S = V(G\bar{G}) \setminus N_{G\bar{G}}[\bar{v}]$ and so $G\bar{G}[S] = P_n$, yielding $I^s(\bar{v}) = 1 + \beta(P_n) = (n+3)/2$.

If a vertex $\bar{v}$ of $\bar{P}_n$ in $G\bar{G}$ is a vertex adjacent to an endvertex of P_n , then $\bar{v}$ all dominates the vertices of $N_{G\bar{G}}[\bar{v}]$ including one of the endvertices $\bar{u}$ of P_n in $\bar{P}_n$. But this vertex is obviously not strong dominated since $deg_{G\bar{G}}(\bar{v}) = n-2 < deg_{G\bar{G}}(u) = n-1$. To strong dominate the vertex $\bar{u}$, take the other endvertex of P_n in $\bar{P}_n$ to $I^s(\bar{v})$ -set. Henceforth, the induced subgraph for the remaining nondominated vertices is the path P_{n-2} , yielding $I^s(\bar{v}) = 2 + \beta(P_{n-2}) = (n+3)/2$.

If a vertex $\bar{v}$ of $\bar{P}_n$ in $G\bar{G}$ is neither an endvertex nor a vertex adjacent to an endvertex in P_n, since the endvertices of P_n in $\bar{P}_n$ are adjacent to vertex $\bar{v}$ in $\bar{P}_n$, then $I^s(\bar{v})$-set includes those two vertices. However, endvertices of P_n in $\bar{P}_n$ are the only vertices of $G\bar{G}$ with maximum degree. Therefore, it is impossible to independent strong dominate the endvertices of P_n in $\bar{P}_n$ with any other vertex of $G\bar{G}$. Hence, $I^s(\bar{v}) = 0$.

Now consider the vertices of P_n in $G\bar{G}$ which are labeled sequentially as $v_0, v_1, ..., v_{n-1}$.

If v_i $(i = 0 \; or \; n-1)$ is an endvertex of P_n in $G\bar{G}$, then v_i cannot strong dominate its neighbors, that is, $N_{G\bar{G}}(v_i) = \begin{cases} \{\bar{v}_i, v_{i+1}\}, & if \; i = 0; \\ \{\bar{v}_i, v_{i-1}\}, & ,if \; i = n-1. \end{cases}$ Since $deg_{G\bar{G}}(\bar{v}_i) = \Delta(G\bar{G})$, vertex $\bar{v}_{i-1}$, if $i = 0$; vertex $\bar{v}_{i+1}$, if $i = n-1$ must be taken to $I^s(v_i)$ -set to strong dominate $\bar{v}_i$, where both $i-1$ and $i+1$ are taken modulo

n. Let $S = \begin{cases} V(G\bar{G}) \setminus \{N_{G\bar{G}}[v_i] \cup N_{G\bar{G}}[\bar{v}_{i-1}]\}, \ if \ i = 0; \\ V(G\bar{G}) \setminus \{N_{G\bar{G}}[v_i] \cup N_{G\bar{G}}[\bar{v}_{i+1}]\}, if \ i = n-1. \end{cases}$ by taking $i-1$ and $i+1$ modulo n. Hence, $G\bar{G}[S] = P_{n-2}$, yielding $I^s(v_i) = 2 + \beta(P_{n-2}) = (n+3)/2$.

If v_i $(1 \leq i \ \leq \lfloor n/2 \rfloor)$ is not an endvertex of P_n, then $I^s(v_i)$-set should somehow include one of the endvertices of P_n in $\bar{P}_n$, vertex $\bar{v}_0$ or $\bar{v}_{n-1}$ since those vertices have the maximum degree in $G\bar{G}$. Take the vertex $\bar{v}_{n-1}$ to $I^s(v_i)$-set. Then, let $S = V(G\bar{G}) \setminus \{N_{G\bar{G}}[v_i] \cup N_{G\bar{G}}[\bar{v}_{n-1}]\}$. Hence, we have $G\bar{G}[S] = P_{i-1} \cup P_{n-i-2}$, yielding $I^s(v_i) = 2 + \beta(P_{i-1} \cup P_{n-i-2}) = 2 + \beta(P_{i-1}) + \beta(P_{n-i-2}) = \begin{cases} (n+3)/2 \ , if \ i \ is \ even; \\ (n+1)/2, if \ i \ is \ odd. \end{cases}$.

As a consequence, for n is odd, $I^s(G\bar{G}) = min\,\{(n+3)/2, 0, (n+1)/2\} = 0$.

Case 2. n is even;

If a vertex $\bar{v}$ of $\bar{P}_n$ in $G\bar{G}$ is an endvertex of P_n , then similar to $Case$ 1, $I^s(\bar{v}) = 1 + \beta(P_n) = (n+2)/2$.

For a vertex $\bar{v}$ of $\bar{P}_n$ in $G\bar{G}$ is a vertex adjacent to an endvertex of P_n in $\bar{P}_n$, then similar to $Case$ 1, $I^s(\bar{v}) = 2 + \beta(P_{n-2}) = (n+2)/2$.

If a vertex $\bar{v}$ of $\bar{P}_n$ in $G\bar{G}$ is neither an endvertex nor a vertex adjacent to an endvertex in P_n , then similar to $Case$ 1, $I^s(\bar{v}) = 0$.

Now consider the vertices of P_n in $G\bar{G}$ which are labeled sequentially as $v_0, v_1, ..., v_{n-1}$.
If v_i $(i = 0 \ or \ n-1)$ is an endvertex of P_n in $G\bar{G}$, then similar to $Case$ 1, $I^s(v_i) = 2 + \beta(P_{n-2}) = (n+2)/2$.
If v_i $(1 \leq i < n/2)$ is not an endvertex of P_n , then similar to $Case$ 1, $I^s(v_i) = (n+2)/2$.

Consequently, for n is even, $I^s(G\bar{G}) = min\,\{(n+2)/2, 0\} = 0$. Thus, the proof holds. □

2.3 Small values

Theorem 2.5. *If a vertex v has eccentricity one in G , then $I^s(v) = 1$.*

Proof. If there exists a vertex v with eccentricity one in a graph G , then this implies that this vertex is attached to all other $n-1$ vertices of G yielding $deg_G(v) = n-1 = \Delta(G)$. Hence, vertex v can independent strong dominate V. Thus, the proof holds. □

Theorem 2.6. *Let G be a graph of order n . If G has a vertex with eccentricity one, then $I^s(G)$ is either 1 or 0.*

Proof. If $G = K_n$, then $\forall v \in V$ $I^s(v) = 1$ and so $I^s(G) = 1$. Otherwise, there exist at least two vertices that cannot strong dominate a vertex with eccentricity one and so with $I^s = 0$ yielding $I^s(G) = 0$. □

Theorem 2.7. *Let G be a graph of order $n > 1$. If G has a unique vertex with eccentricity one, then $I^s(G\bar{G}) = 0$.*

Proof. If G has only one vertex v with eccentricity one in G, then in $G\bar{G}$ since $deg_{G\bar{G}}(v) = n > deg_{G\bar{G}}(u)$ where $u \in N_{G\bar{G}}(v)$, vertex v cannot be strong dominated by any of the vertices of $N_{G\bar{G}}(v)$ for an $I^s_{G\bar{G}}(u)$ -set yielding $I^s_{G\bar{G}}(u) = 0$ and $I^s(G\bar{G}) = 0$. □

Theorem 2.8. *For a graph G of order n, if either G or $\bar{G}$ has diameter one, then $I^s(G\bar{G}) = n$.*

Proof. If either G or $\bar{G}$ has diameter one, then $G\bar{G}$ is the corona $K_n \circ K_1$. Therefore, by Theorem 2.4 (a), the proof is immediate. □

Theorem 2.9. *For a graph G of order n and its complementary prism $G\bar{G}$, $I^s(G\bar{G}) = 1$ if and only if $n = 1$.*

Proof. The sufficiency is immediate since if $n = 1$, then $G\bar{G} = K_2$. Thus, by Theorem 2.4 (a), $I^s(G\bar{G}) = 1$. Now, suppose that $I^s(G\bar{G}) = 1$. This implies that there exists at least one vertex in $G\bar{G}$ that has $I^s(v) = 1$ or has eccentricity one and can independent strong dominate $V(G\bar{G})$. By the structure of complementary prisms, there is a perfect matching between the same labeled corresponding vertices of G and $\bar{G}$. Therefore, the equality of $I^s_{G\bar{G}}(v) = 1$ can only be possible when $G = K_1$. This establishes the necessity. □

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Affiliations

ZEYNEP NIHAN BERBERLER
ADDRESS: Dokuz Eylul University, Dept. of Computer Science, 35160, İzmir-Turkey.
E-MAIL: zeynep.berberler@deu.edu.tr
ORCID ID : 0000 − 0001 − 9179 − 3648