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A Condition for Classical Elastic Curves on Surface

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Abstract
In this paper, we consider two fixed points p to q on a Riemannian surface M in 3-dimensional Euclidean space. We obtain a condition for classical elastic curves with in the family of all curves from p to q on M. We also prove that this condition can be expressed in terms of the curvature functions. The condition is realized for curves whose geodesic and normal curvature functions are both constant.

Keywords: *Energy, Energy of a unit vector field, Elastica.*

AMS Subject Classification (2010): *Primary: 53A04, ; Secondary: 53C20.*

1. Introduction

The problem of the elastic rod which was proposed by Bernoulli is one of the most classical topics in variational calculus. As regards Bernoulli's mathematical idealization, all kinds of elastic curves minimize the total squared curvature among curves of the same length and first order boundary data [1]. Like many of the problems explored by mathematicians of this era, the formal investigation of the elastica was motivated by a physical situation. The Euler-Bernoulli treatment of the elastica transformed a physics problem into one in mathematics. Examining the bending energy of a physical elastic rod was replaced by investigating the total squared curvature of a regular curve [2]. The total squared curvature functional has emerged as a useful quantity in the study of geodesics and the closed thin elastic rod is often used as a model for the DNA molecule [3].

One of the earliest approach on elastica yields prolific consequences on equilibrium of moments which constitute elemantary principle of statics. Further, it is seen that elastica gives a natural solution for the variotional problem which deal with the minimizing of bending energy of the elastic curve. Later, the equivalance between the motion of the simple pendulum and fundamental differential equation of elastica was investigated. Recently, numerical computation implemented on the elastica used to develop math- ematical spline theory [4].

The volume of unit vector fields has been studied by [5], [6], and [7] among other scientists. They define the volume of unit vector field X as the volume of the submanifold of the unit tangent bundle defined by $X(M)$. In [8], the energy of a unit vector field on a Riemannian manifold M is defined as the energy of the mapping $X : M \to T^1M$, where the unit tangent bundle T^1M is equipped with the restriction of the Sasaki metric on TM. The general references [9] and [10] are the classical notation of curve and of surface theory. In [11] author calculated the energy of the Frenet vector fields in R^n, it was shown that the energy of the velocity vector field was $\mathcal{E}(V_1(s)) = \frac{1}{2}\int_a^s k_1^2(u)du$.

In this paper, similar to R^n space, we give a condition for the elastica by defining the surface curve to be elastic. Finally, we give an example to elastic curve on the cylinder.

Definition 1.1 Let M be a surface in R^3 . If p is a point of M , then for each tangent vector v to M at p, let

$$S_p(v) = -\nabla_v Z$$

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where Z is a unit normal vector field on a neighborhood of p in M . S_p is called the shape operator of M at p (derived from Z).

Definition 1.2 A regular curve α in $M \subset R^3$ is a principal curve (or line of curvature) provided that the velocity $\alpha^{'}$ of α always points in a principal direction.

Definition 1.3 A regular curve α in $M \subset R^3$ is a geodesic of M provided its acceleration $\alpha^{''}$ is always normal to M.

Definition 1.4 Let α be a unit-speed curve in $M \subset R^3$. Instead of the Frenet frame field on α, consider the frame field $\{T, Y, Z\}$, where T is the unit tangent of α, Z is the surface normal restricted to α, and , $Y = Z \times T$ then the following relations are valid.

$$T^{'} = K_g Y + K_n Z \tag{1.1}$$
$$Y^{'} = -K_g T + K_t Z \tag{1.2}$$
$$Z^{'} = -K_n T - K_t Y. \tag{1.3}$$

where $K_n =< S(T), T >$ is the normal curvature $K_n(T)$ of in the T direction, and $K_t =< S(T), Y >$ is the torsion function. The new function K_g is called the geodesic curvature of α.

Definition 1.5 Let α be a unit-speed curve in $M \subset R^3$.

$$Then\ \alpha\ is\ geodesic \Leftrightarrow K_g = 0, \tag{1.4}$$

$$then\ \alpha\ is\ asymptotic \Leftrightarrow K_n = 0. \tag{1.5}$$

Definition 1.6 A curve segment is the portion of a curve defined in a closed interval.

Definition 1.7 Let V and W be two vector fields on M, and Z be a normal vector field on M. We may then decompose $D_V W$ as

$$D_V W = \nabla_V W + II(V, W)Z$$

where $\nabla_V W$ and $II(V, W)$ are the tangential component and the normal component of $D_V W$, respectively. This decomposition is known as the *Gauss equation.*

Definition 1.8 For $\eta_1, \eta_2 \in T_\xi(T^1M)$ define

$$g_S(\eta_1, \eta_2) =< d\pi(\eta_1), d\pi(\eta_2) > + < K(\eta_1), K(\eta_2) > . \tag{1.6}$$

This gives a Riemannian metric on TM. Recall that g_S is called *the Sasaki metric*. The metric g_s makes the projection $\pi : T^1M \to M$ a Riemannian submersion [12] and [13].

Proposition 1.1 The connection map $K : T(T^1M) \to T^1M$ verifies the following conditions.
1) $\pi \circ K = \pi \circ d\pi$ and $\pi \circ K = \pi \circ \widetilde{\pi}$, where $\widetilde{\pi} : T(T^1M) \to T^1M$ is the tangent bundle projection.
2) For $\omega \in T_x M$ and a section $\xi : M \to T^1M$, we have

$$K(d\xi(\omega)) = \nabla_\omega \xi$$

where ∇ is the Levi-Civita covariant derivative [12].

Definition 1.9. The energy of a differentiable map
$f : (M, <,>) \to (N, h)$ between Riemannian manifolds is given by

$$\mathcal{E}(f) = \frac{1}{2} \int_M (\sum_{a=1}^{n} h(df(e_a), df(e_a))\upsilon \tag{1.7}$$

where υ is the canonical volume form in M and $\{e_a\}$ is a local basis of the tangent space (see [8, 14], for example).

Let $C^{\infty}(M;N)$ denote the space of all smooth maps from M to N. A map $f: M \to N$ is said to be harmonic if it is an extremal (i.e., critical point) of the energy functional $E(.;D): C^{\infty}(M;N) \to R$ for any compact domain D.

By a (smooth) variation of f we mean a smooth map $f: M \times (-\epsilon, \epsilon) \to N,\ \ (x,t) \to f_t(x)\ \ (\epsilon > 0)$ such that $f_0 = f$. We can think of $\{f_t\}$ as a family of smooth mappings which depends 'smoothly' on a parameter $t \in (-\epsilon, \epsilon)$.

***Definition* 1.10.** A smooth map $f:(M,g) \to (N,h)$ is said to be harmonic if

$$\frac{d}{dt}\mathcal{E}(f_t;D)|_{t=0} = o \tag{1.8}$$

where $E(f;D) = \frac{1}{2}\int_D(\sum_{a=1}^{n} h(df(e_a), df(e_a))v_g$, for all compact domains D and all smooth variations f_t of f supported in D [15].

***Definition* 1.11** Let α be a regular curve defined on any fixed interval [a; b]. Elastica is defined for the curve α in R^n over the each point on a fixed interval [a; b] as a minimizer of the bending energy:

$$\mathcal{E}_B = \frac{1}{2}\int_a^b k_1^2(s)ds,$$

with some boundary conditions [15] and [16].

2. Classical Elastic Curves on Surface

***Definition* 2.1.** A regular curve on surface is called elastica if the velocity vector field of the curve is harmonic.

Let $\varphi: U \subset R^2 \to R^3,\ \varphi(U) \subset M,\ \varphi(U) = (\varphi_1(u,v), \varphi_2(u,v), \varphi_3(u,v))$ and $\varphi(u,v)$ be a local parametrization of surface M in R^3.

Theorem 2.1. Let α be unit speed curve on surface M and $\alpha(a) = p,\ \ \alpha(b) = q$. If α is classical elastic curve, then the following equation is satisfied,

$$\int_a^b \lambda(s)(K_g(s)K_g^{'}(s) + K_n(s)K_n^{'}(s))ds = 0 \tag{2.1}$$

where K_n, K_g are the normal curvature function and the geodesic curvature function of α and λ is the real-valued function on $[a,b]$.

Proof. Let (I,α) be a parametric pair for a unit speed curve C on $\varphi(U) \subset M$ and $\alpha = \varphi \circ \gamma,\ \gamma = (\gamma_1, \gamma_2)$. Let $\{T, Y, Z\}$ be the Frenet frame field on α in M.
We define λ and v_i functions to get a fixed two points on the surface and a collection of curves passing through these two points.
There existe $\lambda : [a,b] \subset I \to R,\ \lambda(s) = (s-a)(b-s),\ \lambda(a) = 0,\ \lambda(b) = 0$ and $\lambda(s) \neq 0$ for all $s \in (a,b)$, of class C^2. Since $\{\varphi_1(\gamma(s)), \varphi_2(\gamma(s))\}$ is a local basis of the tangent space, where φ_1, φ_2 are first-order partial derivatives, we have

$$\lambda(s)T(s) = \Sigma_{i=1}^{2} v_i(s)\varphi_i(\gamma(s));\ \ where\ \ v_i : [a,b] \to R. \tag{2.2}$$

Let the collection of curve be

$$\alpha^k(s) = \varphi(\gamma_1(s) + kv_1(s), \gamma_2(s) + kv_2(s)). \tag{2.3}$$

for $k = 0,\ \ \alpha^0(s) = \alpha(s)$ and

$$(\varphi^{-1} \circ \alpha^k)(s) = \gamma^k(s) = (\gamma_1(s) + kv_1(s), \gamma_2(s) + kv_2(s)).$$

From (10) we get $\lambda(a)T(a) = \Sigma_{i=1}^{2} v_i(a)\varphi_i(\gamma(a))$. Since $\lambda(a) = 0$ we have $v_1(a) = v_2(a) = 0$ and

$$\gamma^k(a) = (\gamma_1(a) + kv_1(a), \gamma_2(a) + kv_2(a) = (\gamma_1(a), \gamma_2(a)) = \gamma(a).$$

Similarly we get $v_1(b) = v_2(b) = 0$ and $\gamma^k(b) = \gamma(b)$. Using these results in (11) we obtain
$\alpha^k(a) = (\varphi \circ \gamma^k)(a) = \alpha(a) = p$ and $\alpha^k(b) = (\varphi \circ \gamma^k)(b) = \alpha(b) = q$.
These results show that α^k is curve segment from p to q on M. Take this collection $\alpha^k(s) = \alpha(s,k)$ for all curve. The expression for energy of velocity vector field T_k of α^k from p to q on M becomes $\mathcal{E}(T_k)$.
On the other hand, let TC_k be the tangent bundle. So we have $T_k : C_k \to TC_k = \bigcup_{s\in I} T_{\alpha^k(s)}C_k.$, where $C_k = \alpha^k(I)$ and $T_{\alpha^k(s)}C_k$ denotes generated by T_k. Let $\pi : TC_k \to C_k$ be the bundle projection. By using equation (7) we obtain that the energy of T_k is

$$\mathcal{E}(T_k) = \frac{1}{2}\int_a^b g_{\mathcal{S}}(dT_k(T_k(\alpha(s,k)), dT_k(T_k(\alpha(s,k)))ds \tag{2.4}$$

where ds is the element of arc length. From (6) we have

$$g_{\mathcal{S}}(dT_k(T_k), dT_k(T_k)) =< d\pi(dT_k(T_k)), d\pi(dT_k(T_k)) > + < K(dT_k(T_k)), K(dT_k(T_k)) > .$$

Since T_k is a section we have $d(\pi)\circ d(T_k) = d(\pi\circ T_k) = d(id_{C_k}) = id_{TC_k}$ we also have by Proposition 1.1 that $K(dT_k(T_k)) = \nabla_{T_k}T_k = T_k^{'}$, then

$$g_{\mathcal{S}}(dT_k(T), dT_k(T)) =< T_k, T_k > + < T_k^{'}, T_k^{'} > .$$

Using these results in (12) we get

$$\mathcal{E}(T_k) = \frac{1}{2}\int_a^b (< T_k, T_k > + < T_k^{'}, T_k^{'} >)ds \tag{2.5}$$

where $T_k = \frac{1}{w(s,k)}\frac{\partial\alpha}{\partial s}(s,k);\ \ w(s,k) = \sqrt{< \frac{\partial\alpha}{\partial s}(s,k),\ \ \frac{\partial\alpha}{\partial s}(s,k) >},\ \ T_k^{'} = \frac{\partial T_k}{\partial s}$.
From (13) we obtain:

$$\frac{\partial\mathcal{E}(T_k)}{\partial k} = \frac{1}{2}[\int_a^b \frac{\partial}{\partial k}[(< T_k, T_k > + < \frac{\partial T_k}{\partial s}, \frac{\partial T_k}{\partial s} >]ds.$$

Since $< T_k, T_k >= 1$ we have $\frac{\partial}{\partial k} < T_k, T_k >= 0$ and by using equation (1), we get

$$\frac{\partial\mathcal{E}(T_k)}{\partial k} = \frac{1}{2}[\int_a^b \frac{\partial}{\partial k} < \frac{\partial T_k}{\partial s}, \frac{\partial T_k}{\partial s} >]ds = \frac{1}{2}\int_a^b \frac{\partial}{\partial k}(K_g^2 + K_n^2)ds. \tag{2.6}$$

Since α is classical elastic curve, by definition 2.1 and (8) we have

$$\frac{\partial\mathcal{E}(T_k)}{\partial k}_{|k=0} = (\frac{1}{2}\int_a^b \frac{\partial}{\partial k}(K_g^2 + K_n^2)ds)_{|k=0} = 0. \tag{2.7}$$

from (14) we have

$$\frac{\partial\mathcal{E}(T_k)}{\partial k} = \frac{1}{2}\int_a^b \frac{\partial}{\partial k} < \frac{\partial T_k}{\partial s}, \frac{\partial T_k}{\partial s} > ds = \int_a^b < \frac{\partial^2 T_k}{\partial s\partial k}, \frac{\partial T_k}{\partial s} > ds. \tag{2.8}$$

We can write

$$\frac{\partial}{\partial s} < \frac{\partial T_k}{\partial k}, \frac{\partial T_k}{\partial s} >=< \frac{\partial^2 T_k}{\partial s\partial k}, \frac{\partial T_k}{\partial s} > + < \frac{\partial T_k}{\partial k}, \frac{\partial^2 T_k}{\partial s^2} > .$$

Thus, we can deduce,

$$< \frac{\partial^2 T_k}{\partial s\partial k}, \frac{\partial T_k}{\partial s} >= \frac{\partial}{\partial s} < \frac{\partial T_k}{\partial k}, \frac{\partial T_k}{\partial s} > - < \frac{\partial T_k}{\partial k}, \frac{\partial^2 T_k}{\partial s^2} > \tag{2.9}$$

Substituting (17) in (16), for, $k=0$,

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}_{|k=0} = \int_a^b [\frac{\partial}{\partial s} < \frac{\partial T_k}{\partial k}(s,0), \frac{\partial T_k}{\partial s}(s,0) > - < \frac{\partial T_k}{\partial k}(s,0), \frac{\partial^2 T_k}{\partial s^2}(s,0) >]ds$$

and

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}_{|k=0} =< \frac{\partial T_k}{\partial k}(s,0), \frac{\partial T_k}{\partial s}(s,0) >|_a^b - \int_a^b < \frac{\partial T_k}{\partial k}(s,0), \frac{\partial^2 T_k}{\partial s^2}(s,0) > ds. \tag{2.10}$$

From (10), (11) we obtain,

$$\frac{\partial \alpha}{\partial k}(s,k)_{|k=0} = \lambda(s)T(s). \tag{2.11}$$

and

$$\frac{\partial \alpha}{\partial s}(s,0) = \alpha^{'}(s) = T(s) = T_k(s.0). \tag{2.12}$$

Now we calculate the partial derivatives of (20) with respect to s and k; using (1), we get

$$\frac{\partial T_k}{\partial s}(s,0) = \frac{\partial^2 \alpha}{\partial s^2}(s,0) = \alpha^{''}(s) = T^{'}(s) = K_g(s)Y(s) + K_n(s)Z(s) \tag{2.13}$$

and

$$\frac{\partial T_k}{\partial k}(s,k) = \frac{\partial^2 \alpha}{\partial s \partial k}(s,k) = \frac{\partial^2 \alpha}{\partial k \partial s}(s,k).$$

So, (19) gives us that

$$\frac{\partial T}{\partial k}(s,k)_{|_{k=0}} = \frac{\partial T}{\partial k}(s,0) = \lambda^{'}(s)T(s) + \lambda(s)K_g(s)Y(s) + \lambda(s)K_n(s)Z(s). \tag{2.14}$$

Therefore, (21) and (22) gives us that

$$< \frac{\partial T_k}{\partial k}(s,0), \frac{\partial T_k}{\partial s}(s,0) >= \lambda(s)(K_g^2(s) + K_n^2(s)).$$

Considering the candidate function $\lambda(a) = \lambda(b) = 0$, we get:

$$< \frac{\partial T_k}{\partial k}(s,0), \frac{\partial T_k}{\partial s}(s,0) >|_a^b= \lambda(b)(K_g^2(b) + K_n^2(b)) - \lambda(a)(K_g^2(b) + K_n^2(a)) = 0. \tag{2.15}$$

Now we calculate the derivative (21) with respect to s,

$$\frac{\partial^2 T}{\partial s^2}(s,0) = K_g^{'}(s)Y(s) + K_g(s)Y^{'}(s) + K_n^{'}(s)Z(s) + K_n(s)Z^{'}(s)$$

By (2) and (3) we have

$$\frac{\partial^2 T}{\partial s^2}(s,0) = -(K_g^2(s) + K_n^2(s))T(s) + (K_g^{'}(s) - K_n(s)K_t(s))Y(s) + (K_g(s)K_t(s) + K_n^{'}(s))Z(s). \tag{2.16}$$

So (22) and (24) given us that

$$< \frac{\partial T_k}{\partial k}(s,0), \frac{\partial^2 T_k}{\partial s^2}(s,0) >= -\lambda^{'}(s)(K_g^2(s) + K_n^2(s)) + \lambda(s)(K_g(s)K_g^{'}(s) + K_n(s)K_n^{'}(s))$$

and

$$< \frac{\partial T_k}{\partial k}(s,0), \frac{\partial^2 T_k}{\partial s^2}(s,0) >= [-\lambda(s)(K_g^2(s)+K_n^2(s))]^{'} + 3\lambda(s)(K_g(s)K_g^{'}(s)+K_n(s)K_n^{'}(s)) \tag{2.17}$$

Using (23) and (25) in (18), yields that

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}_{|k=0} = -\int_a^b ([-\lambda(s)(K_g^2(s)+K_n^2(s))]^{'} + 3\lambda(s)(K_g(s)K_g^{'}(s)+K_n(s)K_n^{'}(s))ds$$

and

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}_{|k=0} = (\lambda(s)(K_g^2(s)+K_n^2(s))|_a^b - 3\int_a^b \lambda(s)(K_g(s)K_g^{'}(s)+K_n(s)K_n^{'}(s))ds$$

We are looking the candidate function $\lambda(a)=\lambda(b)=0$ which gives

$$(\lambda(s)(K_g^2(s)+K_n^2(s))|_a^b = 0$$

from (15), we have

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}_{|k=0} = -3\int_a^b \lambda(s)(K_g(s)K_g^{'}(s)+K_n(s)K_n^{'}(s))ds = 0.$$

This completes the proof of the theorem.
From equations (4)and (5), if α is both principal and asymptotic or the normal curvature function and the geodesic curvature function of α are constant, then is satisfy (9) equation.

Example 1. Let $\varphi : [-\pi,\pi]\times R \to R^3$, $\varphi(\theta,h) = (cos\theta, sin\theta, h,)$ and $\beta(s) = (cos(coss), sin(coss), sins)$; $\beta(-\pi) = p$, $\beta(\pi) = q$. If we can choose $\lambda : [-\pi,\pi] \to R$, $\lambda(s) = \pi^2 - s^2$ then $\lambda(-\pi) = 0$ $\lambda(\pi) = 0$ and $\lambda(s) \neq 0$ for all $s \in (-\pi,\pi)$. We calculate;

$$T(s) = (sins.sin(coss), -sins.cos(coss), Coss),$$
$$Z(s) = (cos(coss), sin(coss), 0),$$
$$Y(s) = (coss.sin(coss), -coss.cos(coss), -sins),$$
$$K_g(s) = 1,\ \ K_n(s) = -sin^2 s$$

and

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}_{|k=0} = 6\int_{-\pi}^{\pi} (\pi^2 - s^2)sin^2 s.sins.coss.ds = 0.$$

Thus β is elastica on the cylinder, Figure 1.

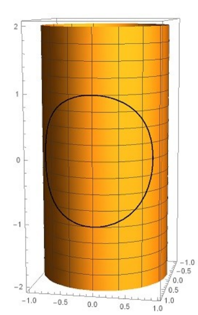

Figure 1

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