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On the hyper-gamma function

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Abstract
In this paper, we introduce a new generalization for the gamma function as hyper-gamma function. Some identities and integral representation are obtained for the this new generalization.

Keywords: *Gamma function; generalization; hyper-gamma function.*

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1. Introduction

There are a few special functions in mathematics that have particular significance and many applications in many branches such as probability, statistics, physics, engineering, and other mathematical sciences. One of those functions is the Euler's gamma function. For $x > 0$, the Euler's gamma function is defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

For extensions of the gamma function to complex variables and for the basic properties see [12, p. 235-264]. The recursion formula for the gamma function

$$\Gamma(x+1) = x\Gamma(x)$$

is well known [12] and this yields

$$\Gamma(x+n) = (x)_n \Gamma(x)$$

where $(x)_n$ is the Pochhammer symbol defined as

$$(x)_0 = 1 \text{ and } (x)_n = x(x+1)(x+2)\cdots(x+n-1).$$

The gamma function has very extensive literature, especially; recently, numerous papers have been published concerning with inequalities for the gamma and related functions [1,2,3,4]. Anderson et all. [3] obtained

$$\lim_{x\to\infty} \frac{\log\Gamma\left(1+\frac{x}{2}\right)}{x\log x} = \frac{1}{2}. \tag{1.1}$$

Anderson and Qiu [2] proved that

$$\lim_{x\to\infty} \frac{\log\Gamma(x)}{(x-1)\log(x-1)} = 1 \tag{1.2}$$

and for $x > 1$,

$$x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1} \tag{1.3}$$

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where γ is the Euler–Mascheroni constant.

Gamma and related functions have some generalizations [5,6,7,8]. For example, Chaudhry and Zubair [5] have introduced the following extension for the gamma function

$$\Gamma_p(x) = \int_0^\infty e^{-t-pt^{-1}} t^{x-1} dt$$

where $Re(p) > 0$.

In this paper, we introduce a new generalization for the gamma function as nth hyper-gamma function of order r defined as

$$\Gamma_n^{(r)}(s) = \sum_{p=0}^{n} \Gamma_p^{(r-1)}(s), \qquad (r \geq 1, n \geq 0 \text{ and } Re(s) > 0)$$

where $\Gamma_0^{(n)}(s) = \Gamma(s)$, $\Gamma_n^{(0)}(s) = \Gamma(s+n)$ and $\Gamma(s)$ is the classical gamma function. We give the name "hyper-gamma function" to our generalization because its representation is similar to the representation of hyperharmonic number see [9,10].

In section 2, we study some properties of $\Gamma_n^{(r)}(s)$. Moreover, we give two limits and an inequality concerning with $\Gamma_n^{(r)}(s)$.

2. The main results

Theorem 2.1. *Let $\Gamma_n^{(r)}(s)$ be nth hyper-gamma function of order r. If $r \geq 1$ and $m \geq 0$, then*

$$\Gamma_n^{(m+r)}(s) = \sum_{p=0}^{n} \binom{n+r-p-1}{r-1} \Gamma_p^{(m)}(s).$$

Proof. Let (a_n) and (a^n) be two real initial sequences. The entries a_n^k corresponding to these sequences are determined recursively by the formulas

$$\begin{aligned} a_n^0 &= a_n, \quad a_0^n = a^n \qquad (n \geq 0) \\ a_n^k &= a_n^{k-1} + a_{n-1}^k \qquad (n \geq 1, k \geq 1) \end{aligned}$$

The entries a_n^k have the following symmetric relation[10, relation 2]:

$$a_n^k = \sum_{i=1}^{k} \binom{n+k-i-1}{n-1} a_0^i + \sum_{t=1}^{n} \binom{n+k-t-1}{k-1} a_t^0. \tag{2.1}$$

It is clear that $\Gamma_n^{(r)}(s)$ has the recurrence relation as follows: $\Gamma_n^{(r)}(s) = \Gamma_n^{(r-1)}(s) + \Gamma_{n-1}^{(r)}(s)$. Hence, If we select $a_n^0 = \Gamma_n^{(m)}(s)$ and $a_0^n = \Gamma_0^{(m+n)}(s) = \Gamma(s)$, $n \geq 1$, then $a_n^r = \Gamma_n^{(m+r)}(s)$ and from relation (2.1) we have

$$\begin{aligned} \Gamma_n^{(m+r)}(s) &= \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} \Gamma(s) + \sum_{t=1}^{n} \binom{n+r-t-1}{r-1} \Gamma_t^{(m)}(s) \\ &= \Gamma(s) \sum_{i=0}^{r-1} \binom{n+r-i-2}{n-1} + \sum_{t=0}^{n-1} \binom{n+r-t-2}{r-1} \Gamma_{t+1}^{(m)}(s). \end{aligned} \tag{2.2}$$

With selections $k = r-i-1$ and $b = n-t-1$, the Eq. (2.2) is written as

$$\Gamma_n^{(m+r)}(s) = \Gamma(s) \sum_{k=0}^{r-1} \binom{n+k-1}{n-1} + \sum_{b=0}^{n-1} \binom{b+r-1}{r-1} \Gamma_{n-b}^{(m)}(s).$$

From the following nice combinatorial identity [11, p. 160]

$$\sum_{k=0}^{r-1} \binom{n+k-1}{n-1} = \binom{n+r-1}{n}, \tag{2.3}$$

we have

$$\begin{aligned}\Gamma_n^{(m+r)}(s) &= \binom{n+r-1}{n}\Gamma(s)+\sum_{b=0}^{n-1}\binom{b+r-1}{r-1}\Gamma_{n-b}^{(m)}(s)\\ &= \sum_{b=0}^{n}\binom{b+r-1}{r-1}\Gamma_{n-b}^{(m)}(s)\\ &= \sum_{p=0}^{n}\binom{n+r-p-1}{r-1}\Gamma_{p}^{(m)}(s)\end{aligned}$$

where $p=n-b$. Thus the proof is completed. □

Corollary 2.1. *The following identities hold*

$$i)\ \Gamma_n^{(r)}(s)=\sum_{p=0}^{n}\left[\binom{n+r-p-1}{r-1}\Gamma(s+p)\right]. \tag{2.4}$$

$$ii)\ \Gamma_n^{(r)}(s)=\Gamma(s)\sum_{p=0}^{n}\left[\binom{n+r-p-1}{r-1}(s)_p\right] \tag{2.5}$$

where $(s)_p$ denotes the Pochhammer symbol.

$$iii)\ \Gamma_n^{(r)}(1)=\sum_{p=0}^{n}\binom{n+r-p-1}{r-1}p!. \tag{2.6}$$

$$iv)\ \Gamma_n^{(1)}(s)=\Gamma(s)\sum_{p=0}^{n}(s)_p\,.$$

$$v)\ \Gamma_1^{(r)}(s)=(r+s)\,\Gamma(s)\,.$$

$$vi)\ \Gamma_1^{(1)}(1)=2.$$

Theorem 2.2. *The nth hyper-gamma function of order r, $\Gamma_n^{(r)}(s)$, has the following integral representation*

$$\Gamma_n^{(r)}(s)=\int_0^\infty e^{-u^\alpha}u^{\alpha s-1}du$$

where $\alpha=\left[\sum\limits_{p=0}^{n}\binom{n+r-p-1}{r-1}(s)_p\right]^{-1}$.

Proof. Let α be as $\alpha=\left[\sum\limits_{p=0}^{n}\binom{n+r-p-1}{r-1}(s)_p\right]^{-1}$. Then by using the representation in (2.5) of $\Gamma_n^{(r)}(s)$, we obtain

$$\Gamma_n^{(r)}(s)=\alpha^{-1}\Gamma(s)=\alpha^{-1}\int_0^\infty e^{-t}t^{s-1}dt=\int_0^\infty e^{-t}\frac{t^s}{\alpha t}dt.$$

If we make change of variable $t=u^\alpha$, we have $dt=\alpha u^{\alpha-1}du$ and

$$\begin{aligned}\Gamma_n^{(r)}(s) &= \int_0^\infty e^{-u^\alpha}\frac{u^{\alpha s}}{\alpha u^\alpha}\alpha u^{\alpha-1}du\\ &= \int_0^\infty e^{-u^\alpha}u^{\alpha s-1}du.\end{aligned}$$

□

Theorem 2.3. *For the nth hyper-gamma function of order r, $\Gamma_n^{(r)}(s)$, we have*

$$\sum_{k=1}^{r}\Gamma_n^{(k)}(s)=\Gamma_{n+1}^{(r)}(s)-\Gamma(s+n+1).$$

Proof. By using the representation in (2.4) of $\Gamma_n^{(r)}(s)$, we obtain

$$\begin{aligned}\sum_{k=1}^{r}\Gamma_n^{(k)}(s) &= \sum_{k=1}^{r}\sum_{p=0}^{n}\binom{n+k-p-1}{k-1}\Gamma(s+p)\\ &= \sum_{p=0}^{n}\left[\Gamma(s+p)\sum_{k=1}^{r}\binom{n+k-p-1}{k-1}\right].\end{aligned}$$

If we use the nice combinatorial identity in (2.3) we have

$$\begin{aligned}\sum_{k=1}^{r}\Gamma_n^{(k)}(s) &= \sum_{p=0}^{n}\binom{n+r-p}{r-1}\Gamma(s+p)\\ &= \sum_{p=0}^{n+1}\binom{n+r-p}{r-1}\Gamma(s+p)-\Gamma(s+n+1)\\ &= \Gamma_{n+1}^{(r)}(s)-\Gamma(s+n+1).\end{aligned}$$

Thus the proof is completed. □

Theorem 2.4. *For the nth hyper-gamma function of order r, $\Gamma_n^{(r)}(s)$, the following identities hold*

$$\begin{aligned} i)\sum_{p=1}^{n}p\Gamma_p^{(r)}(s) &= n\Gamma_n^{(r+1)}(s)-\Gamma_{n-1}^{(r+2)}(s)\\ ii)\sum_{p=1}^{r}p\Gamma_n^{(p)}(s) &= r\Gamma_{n+1}^{(r)}(s)-\Gamma_{n+2}^{(r-1)}(s)+(n+s)\Gamma(n+s+1).\end{aligned}$$

Proof. *i*) It is clear that

$$\begin{aligned}\sum_{p=1}^{n}p\Gamma_p^{(r)}(s) &= \Gamma_1^{(r)}(s)+2\Gamma_2^{(r)}(s)+3\Gamma_3^{(r)}(s)+\cdots+(n-1)\Gamma_{n-1}^{(r)}(s)+n\Gamma_n^{(r)}(s)\\ &= \Gamma_0^{(r)}(s)+\Gamma_1^{(r)}(s)+\Gamma_2^{(r)}(s)+\cdots+\Gamma_n^{(r)}(s)-\Gamma_0^{(r)}(s)\\ &\quad+\Gamma_0^{(r)}(s)+\Gamma_1^{(r)}(s)+\Gamma_2^{(r)}(s)+\cdots+\Gamma_n^{(r)}(s)-\Gamma_0^{(r)}(s)-\Gamma_1^{(r)}(s)\\ &\quad\vdots\\ &\quad+\Gamma_0^{(r)}(s)+\Gamma_1^{(r)}(s)+\cdots+\Gamma_n^{(r)}(s)-\Gamma_0^{(r)}(s)-\Gamma_1^{(r)}(s)-\cdots-\Gamma_{n-1}^{(r)}(s).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{p=1}^{n}p\Gamma_p^{(r)}(s) &= n\sum_{p=0}^{n}\Gamma_p^{(r)}(s)-\sum_{p=0}^{n-1}\Gamma_p^{(r+1)}(s)\\ &= n\Gamma_n^{(r+1)}(s)-\Gamma_{n-1}^{(r+2)}(s).\end{aligned}$$

The proof of *ii*) is similar to the proof of *i*). □

Theorem 2.5. *For the nth hyper-gamma function of order r, $\Gamma_n^{(r)}(s)$, the following limits hold*

$$i)\ \lim_{s\to\infty}\frac{\log\Gamma_n^{(r)}\left(1+\frac{s}{2}\right)}{s\log s}=\frac{1}{2}$$

$$ii)\ \lim_{s\to\infty}\frac{\log\Gamma_n^{(r)}\left(s\right)}{\left(s-1\right)\log\left(s-1\right)}=1.$$

Proof. *i*) From Corollary 2.1 *ii*), we have

$$\Gamma_n^{(r)}\left(1+\frac{s}{2}\right)=\Gamma\left(1+\frac{s}{2}\right)\sum_{p=0}^{n}\left[\binom{n+r-p-1}{r-1}\left(1+\frac{s}{2}\right)_p\right].$$

Hence

$$\begin{aligned}\lim_{s\to\infty}\frac{\log\Gamma_n^{(r)}\left(1+\frac{s}{2}\right)}{s\log s} &= \lim_{s\to\infty}\frac{\log\left(\Gamma\left(1+\frac{s}{2}\right)\sum_{p=0}^{n}\left[\binom{n+r-p-1}{r-1}\left(1+\frac{s}{2}\right)_p\right]\right)}{s\log s}\\ &= \lim_{s\to\infty}\frac{\log\Gamma\left(1+\frac{s}{2}\right)}{s\log s}+\lim_{s\to\infty}\frac{\log\sum_{p=0}^{n}\left[\binom{n+r-p-1}{r-1}\left(1+\frac{s}{2}\right)_p\right]}{s\log s}.\end{aligned}$$

Since

$$\lim_{s\to\infty}\frac{\log\sum_{p=0}^{n}\left[\binom{n+r-p-1}{r-1}\left(1+\frac{s}{2}\right)_p\right]}{s\log s}=0$$

and from the Eq. (1.1), we obtain

$$\lim_{s\to\infty}\frac{\log\Gamma_n^{(r)}\left(1+\frac{s}{2}\right)}{s\log s}=\frac{1}{2}.$$

The proof of *ii*) is similar to the proof of *i*). □

Theorem 2.6. *For $s>1$, the following inequalities hold*

$$\binom{n+r}{r}s^{(1-\gamma)s-1}<\Gamma_n^{(r)}\left(s\right)<\binom{n+r}{r}\left(s+n\right)^{s+n-1}$$

where γ is the Euler–Mascheroni constant.

Proof. For $s>1$, it is true that

$$\begin{aligned}\sum_{p=0}^{n}\binom{n+r-p-1}{r-1}\Gamma\left(s\right) &\le \Gamma_n^{(r)}\left(s\right)=\sum_{p=0}^{n}\binom{n+r-p-1}{r-1}\Gamma\left(s+p\right)\\ &\le\sum_{p=0}^{n}\binom{n+r-p-1}{r-1}\Gamma\left(s+n\right).\end{aligned}$$

By considering combinatorial identity in (2.3), the last inequalities above are written as

$$\binom{n+r}{r}\Gamma\left(s\right)\le\Gamma_n^{(r)}\left(s\right)\le\binom{n+r}{r}\Gamma\left(s+n\right).$$

By using the inequalities in (1.3) for $x>1$,

$$x^{(1-\gamma)x-1}<\Gamma\left(x\right)<x^{x-1},$$

we have

$$\binom{n+r}{r}s^{(1-\gamma)s-1}<\Gamma_n^{(r)}\left(s\right)<\binom{n+r}{r}\left(s+n\right)^{s+n-1}.$$

□

3. Conclusion

In this work, we define a new generalization for the gamma function and study some properties of this new generalization. We think that our study can be a reference to future researches on the bounds for values of $\Gamma_n^{(r)}(s)$ and relations of other generalizations and functions with $\Gamma_n^{(r)}(s)$.

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