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T_1 Extended Pseudo-Quasi-Semi Metric Spaces

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Abstract
In this paper, we characterize a T_1 extended pseudo-quasi-semi metric space at p and a T_1 extended pseudo-quasi-semi metric space and investigate the relationships between them. Finally, we compare each of T_1 extended pseudo-quasi-semi metric spaces with the usual T_1.

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1. Introduction

When the classical conditions on a metric d on a set X are relaxed by omitting the requirement that $d(x, y) = 0$ implies $x = y$, then d is called a pseudo-metric which is given in [7]. It is well-known that the category of metric spaces and non-expansive maps does not behave well with respect to the formation of infinite products and coproducts. In 1990, J. Adámek and J. Reiterman [2] defined extended pseudo-metric spaces (where an pseudo-metric is allowed to attain the value infinity) in order to solve this problem. In 1931, Wilson [15] introduced quasi-metric spaces (where the condition of symmetry is omitted) which are common in real life.

In 1991, Baran [3] introduced local T_1 separation property in set-based topological categories and then, it is generalized to point free definition of T_1 by using the generic element method of topos theory ([9], [12]). One of the uses of local T_1 separation property is to define the notion of closed subobject of an object of a topological category which is used in the notions of completely regular [5], regular and normal objects in [6]. One of the other uses of T_1 objects is to define completely regular [5], regular and normal objects in [6] in arbitrary topological categories.

In this paper, we characterize a T_1 extended pseudo-quasi-semi metric space at p and a T_1 extended pseudo-quasi-semi metric space and investigate the relationships between them. Finally, we compare each of T_1 extended pseudo-quasi-semi metric spaces with the usual T_1.

2. Preliminaries

Recall, [8], [1] or [14] that a functor $U : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if U is concrete (i.e., faithful and amnestic (i.e., if $U(f) = id$ and f is an isomorphism, then $f = id$)), has small (i.e., sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift. Note also that U has a right adjoint called the indiscrete functor. Recall, in [1] or [14], that an object $X \in \mathcal{E}$ is indiscrete if and only if every map $U(Y) \to U(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is discrete if and only if every map $U(X) \to U(Y)$ lifts to map $X \to Y$ for each object $Y \in \mathcal{E}$.

An extended pseudo-quasi-semi metric space is a pair (X, d), where X is a set $d : X \times X \to [0, \infty]$ is a function fulfills the following condition $d(x, x) = 0$ for all $x \in X$ [10], [11] or [13] .

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A map $f : (X, d) \to (Y, e)$ between extended pseudo-quasi-semi metric spaces is said to be a non-expansive if it fulfills the property $e(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.
The construct of extended pseudo-quasi-semi metric spaces and non-expansive maps is denoted by **pqsMet**. Note that **pqsMet** is a topological category [10], [11] or [13].

2.1 A source $\{f_i : (X, d) \to (X_i, d_i), i \in I\}$ in **pqsMet** is an initial lift if and only if $d = \sup_{i\in I}(d_i \circ (f_i \times f_i))$, i.e., for all $x, y \in X$,

$$d(x, y) = \sup_{i\in I}(d_i(f_i(x), f_i(y)))$$

[10] or [13].

2.2 The discrete extended pseudo-quasi-semi metric structure d_{dis} on X is given by for all $a, b \in X$

$$d_{dis}(a, b) = \begin{cases} 0 & \text{if } a = b \\ \infty & \text{if } a \neq b \end{cases}$$

[10].

3. T_1 Extended Pseudo-Quasi-Semi Metric Spaces

Let B be a set and $p \in B$. Let $B \vee_p B$ be the wedge at p [3], i.e., two disjoint copies of B identified at p. A point $x \in B \vee_p B$ will be denoted by x_1 (x_2) if x is in the first (resp. the second) component of $B \vee_p B$. Note that $p_1 = p_2$. The Skewed p-axis map $S_p : B \vee_p B \to B^2$ is given by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, $\nabla_p : B \vee_p B \to B$ is given by $\nabla_p(x_i) = x$ for $i = 1, 2$ [3].
Let $M = \Delta \subset B^2$ be the diagonal and $B^2 \bigvee_\Delta B^2$, the wedge at Δ, i.e., two distinct copies of B^2 identified along the diagonal, i.e., the result of pushing out Δ along itself [3]. A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1$ ($(x, y)_2$) if (x, y) is in the first (resp. second) component of $B^2 \vee_\Delta B^2$. Clearly $(x, y)_1 = (x, y)_2$ if and only if $x = y$ [3]. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$ [3].

Definition 3.1. Let (X, τ) be a topological space and $p \in X$. If for each point x distinct from p, there exists a neighborhood of p missing x and there exists a neighborhood of x missing p, then (X, τ) is called T_1 at p [3], [4].

Theorem 3.1. *(1) A topological space (X, τ) is called T_1 at p if and only if the initial topology induced from $S_p : X \vee_p X \to (X^2, \tau_*)$ and $\nabla_p : X \vee_p X \to (X, P(X))$ is discrete, where ,τ_* and $P(X)$ is the product topology on X^2 and the discrete topology on X, respectively [4].*
(2) A topological space $[X, \tau)$ is called T_1 if and only if the initial topology induced from $S : X^2 \vee_\Delta X^2 \to (X^3, \tau_*)$ and $\nabla : X^2 \vee_\Delta X^2 \to (X^2, P(X^2))$ is discrete, where ,τ_* and $P(X^2)$ is the product topology on X^3 and the discrete topology on X^2, respectively [4].

Definition 3.2. (cf. [3]) Let $U : \mathcal{E} \to$ SET be topological, X an object in $\mathcal{E}$ and $p \in U(X) = B$.
(1) X is called T_1 at p if the initial lift of the U -source $\{S_p : B \vee_p B \to U(X) = B$ and $\nabla_p : B \vee_p B \to U(D(B)) = B\}$ is discrete, where D is the discrete functor which is a left adjoint to U.
(2)X is called T_1 if the initial lift of the U-source $\{S : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2)) = B^2\}$ is discrete.

Theorem 3.2. *An extended pseudo-quasi-semi metric space (X, d) is T_1 at p if and only if for all $x \in X$ with $x \neq p$, $d(x, p) = \infty = d(p, x)$.*

Proof. Suppose that (X, d) is T_1 at p and for $x \in X$ with $x \neq p$. Let $\pi_i : X^2 \to X$, $i = 1, 2$ be the projection maps and d_{dis} be the discrete extended pseudo-quasi-semi metric structure on X. Since (X, d) is T_1 at p and $x_1 \neq x_2$, by 2.1, 2.2, and Definition 3.2,

$$d_{dis}(\nabla_p(x_1), \nabla_p(x_2)) = d_{dis}(x, x) = 0,$$

$$d(\pi_1 S_p(x_1), \pi_1 S_p(x_2)) = d(x,p),$$
$$d(\pi_2 S_p(x_1), \pi_2 S_p(x_2)) = d(x,x) = 0$$

$$\begin{aligned}\infty &= \sup\{d_{dis}(\nabla_p(x_1), \nabla_p(x_2)), d(\pi_1 S_p(x_1), \pi_1 S_p(x_2)), d(\pi_2 S_p(x_1), \pi_2 S_p(x_2))\} \\ &= \sup\{d(x,p), 0\} \\ &= d(x,p)\end{aligned}$$

and consequently, $d(x,p) = \infty$. Similarly,

$$d_{dis}(\nabla_p(x_2), \nabla_p(x_1)) = 0,$$
$$d(\pi_1 S_p(x_2), \pi_1 S_p(x_1)) = d(p,x),$$
$$d(\pi_2 S_p(x_2), \pi_2 S_p(x_1)) = d(x,x) = 0$$

$$\begin{aligned}\infty &= \sup\{d_{dis}(\nabla_p(x_1), \nabla_p(x_2)), d(\pi_1 S_p(x_1), \pi_1 S_p(x_2)), d(\pi_2 S_p(x_1), \pi_2 S_p(x_2))\} \\ &= \sup\{d(p,x), 0\} \\ &= d(p,x)\end{aligned}$$

and thus, $d(p,x) = \infty$.
Conversely, suppose that for $x \in X$ with $x \neq p$, $d(x,p) = \infty = d(p,x) = \infty$. We need to show that (X,d) is T_1 at p. Let $\bar{d}$ be the extended pseudo-quasi-semi metric structure on $X \vee_p X$ induced by $S_p : X \vee_p X \to (X^2, d^2)$ and $\nabla_p : X \vee_p X \to (X, d_{dis})$, where d^2 and d_{dis} are the product extended pseudo-quasi-semi metric structure on X^2 and the discrete extended pseudo-quasi-semi metric structure on X, respectively.
Let u and v be any points in $X \vee_p X$. If $u = v$, then $\bar{d}(u,v) = 0$.

Suppose that $u \neq v$. If $\nabla_p(u) \neq \nabla_p(v)$, then, by 2.2, $d_{dis}(\nabla_p(u), \nabla_p(v)) = \infty$, and consequently, by 2.1,

$$\bar{d}(u,v) = \sup\{d_{dis}(\nabla_p(u), \nabla_p(v)) = \infty, d(\pi_1 S_p(u), \pi_1 S_p(v)), d(\pi_2 S_p(u), \pi_2 S_p(v))\} = \infty.$$

Suppose that $u \neq v$ and $\nabla_p(u) = x = \nabla_p(v)$ for some $x \in X$ with $x \neq p$. It follows that $u = x_1$ and $v = x_2$ or $u = x_2$ and $v = x_1$. If $u = x_1$ and $v = x_2$, then, by 2.1,

$$d(\pi_1 S_p(u), \pi_1 S_p(v)) = d(x,p),$$
$$d(\pi_2 S_p(u), \pi_2 S_p(v)) = d(x,x) = 0,$$
$$d_{dis}(\nabla_p(u), \nabla_p(v)) = d_{dis}(x,x) = 0$$

$$\begin{aligned}\bar{d}(u,v) &= \sup\{d(\pi_1 S_p(u), \pi_1 S_p(v)), d(\pi_2 S_p(u), \pi_2 S_p(v)), d_{dis}(\nabla_p(u), \nabla_p(v))\} \\ &= \sup\{0, d(x,p)\} = d(x,p) = \infty\end{aligned}$$

by the assumption.

If $u = x_2$ and $v = x_1$, then, by 2.1,

$$d(\pi_1 S_p(u), \pi_1 S_p(v)) = d(p,x),$$
$$d(\pi_2 S_p(u), \pi_2 S_p(v)) = d(x,x) = 0,$$
$$d_{dis}(\nabla_p(u), \nabla_p(v)) = d_{dis}(x,x) = 0$$

$$\begin{aligned}\bar{d}(u,v) &= \sup\{d(\pi_1 S_p(u), \pi_1 S_p(v)), d(\pi_2 S_p(u), \pi_2 S_p(v)), d_{dis}(\nabla_p(u), \nabla_p(v))\} \\ &= \sup\{d(p,x), 0\} = d(p,x) = \infty\end{aligned}$$

since $x \neq p$ and $d(p,x) = \infty$.
Hence, for any points u and v in $X \vee_p X$

$$\bar{d}(u,v) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } u \neq v \end{cases}$$

and consequently, by 2.1, 2.2, and Definition 3.2, (X,d) is T_1 at p. □

Theorem 3.3. *An extended pseudo-quasi-semi metric space* (X, d) *is* T_1 *if and only if* (X, d) *is a discrete extended pseudo-quasi-semi metric space.*

Proof. Suppose that (X, d) is T_1 and for every distinct pair x and y in X , $u = (x, y)_1, v = (x, y)_2$. Note that $u = (x, y)_1$ and $v = (x, y)_2$ are points in the wedge $X^2 \vee_\Delta X^2$ and

$$d(\pi_1 S(u), \pi_1 S(v)) = d(x, x) = 0 = d_{dis}((x, y), (x, y)) = d_{dis}(\bigtriangledown(u), \bigtriangledown(v)),$$

$$d(\pi_2 S(u), \pi_2 S(v)) = d(y, x), d(\pi_3 S(u), \pi_3 S(v)) = d(y, y) = 0.$$

Since (X, d) is T_1 and $u \neq v$, by 2.1, 2.2, and Definition 3.2,

$$\infty = \bar{d}(u, v) = \sup\{d_{dis}(\bigtriangledown(u), \bigtriangledown(v)), d(\pi_i S(u), \pi_i S(v)), i = 1, 2, 3\} = d(y, x).$$

If $u = (x, y)_2, v = (x, y)_1$, then

$$d(\pi_1 S(u), \pi_1 S(v)) = d(x, x) = 0 = d_{dis}((x, y), (x, y)) = d_{dis}(\bigtriangledown(u), \bigtriangledown(v)),$$

$$d(\pi_2 S(u), \pi_2 S(v)) = d(x, y), d(\pi_3 S(u), \pi_3 S(v)) = d(y, y) = 0.$$

Since (X, d) is T_1 and $u \neq v$, by 2.1, 2.2, and Definition 3.2,

$$\infty = \bar{d}(u, v) = \sup\{d_{dis}(\bigtriangledown(u), \bigtriangledown(v)), d(\pi_i S(u), \pi_i S(v)), i = 1, 2, 3\} = d(x, y).$$

Thus, for every distinct pair x and y in X, we have $d(x, y) = \infty = d(y, x)$ and by 2.2, (X, d) is a discrete extended pseudo-quasi-semi metric space.
Conversely, suppose that (X, d) is a discrete extended pseudo-quasi-semi metric space. Let $\bar{d}$ be the initial extended pseudo-quasi-semi metric structure on $X^2 \vee_\Delta X^2$ induced by $S : X^2 \vee_\Delta X^2 \to U((X^3, d^3)) = X^3$ and $\bigtriangledown : X^2 \vee_\Delta X^2 \longrightarrow U((X^2, d_{dis})) = X^2$, where d^3 and d_{dis} are the product extended pseudo-quasi-semi metric structure on X^3 and the discrete extended pseudo-quasi-semi metric structure on X^2, respectively. We need to show that (X, d) is T_1 , i.e., by 2.1, 2.2, and Definition 3.2, we must show that the extended pseudo-quasi-semi metric structure $\bar{d}$ is discrete.
Let u and v be any points in the wedge $X^2 \vee_\Delta X^2$. If $u = v$, then $\bar{d}(u, v) = 0$. Suppose that $u \neq v$. If $\bigtriangledown(u) \neq \bigtriangledown(v)$, then, by 2.2, $d_{dis}(\bigtriangledown(u), \bigtriangledown(v)) = \infty$, and consequently, $\bar{d}(u, v) = \infty$. Suppose that $u \neq v$ and $\bigtriangledown(u) = (x, y) = \bigtriangledown(v)$ for some $(x, y) \in X^2$ with $x \neq y$. Since $u \neq v$, we must have $u = (x, y)_1, v = (x, y)_2$ or $u = (x, y)_2, v = (x, y)_1$. If $u = (x, y)_1$ and $v = (x, y)_2$, then, by 2.1,

$$d(\pi_1 S(u), \pi_1 S(v)) = d(x, x) = 0 = d_{dis}(\bigtriangledown(u), \bigtriangledown(v))$$

$$d(\pi_2 S(u), \pi_2 S(v)) = d(y, x)$$

$$d(\pi_3 S(u), \pi_3 S(v)) = d(y, y) = 0$$

$$\bar{d}(u, v) = \sup\{d(\pi_1 S(u), \pi_1 S(v)), d(\pi_2 S(u), \pi_2 S(v)), d(\pi_3 S(u), \pi_3 S(v))\} = d(y, x) = \infty$$

since $x \neq y$ and d is discrete.
If $u = (x, y)_2, v = (x, y)_1$, then then, by 2.1,

$$d(\pi_1 S(u), \pi_1 S(v)) = d(x, x) = 0 = d_{dis}(\bigtriangledown(u), \bigtriangledown(v)),$$

$$d(\pi_2 S(u), \pi_2 S(v)) = d(x, y),$$

$$d(\pi_3 S(u), \pi_3 S(v)) = d(y, y) = 0$$

$$\bar{d}(u, v) = \sup\{d(\pi_1 S(u), \pi_1 S(v)), d(\pi_2 S(u), \pi_2 S(v)), d(\pi_3 S(u), \pi_3 S(v))\} = d(x, y) = \infty$$

since $x \neq y$ and d is discrete.

Hence, for any points u and v in $X^2 \vee_\Delta X^2$

$$\bar{d}(u, v) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } u \neq v \end{cases}$$

and consequently, by 2.1, 2.2 and Definition 3.2, (X, d) is T_1.

□

Note that every extended pseudo-quasi-semi metric space (X, d) defines a topology denoted by τ_d as in the usual metric space case. Let $x \in X$ and $r > 0$. The set $S(x, r) = \{y \in X : d(x, y) < r\}$ is called a ball in X. Recall that (X, d) is T_1 if and only if the topological space (X, τ_d) is $\mathbf{T}_1$ (we will refer to it as the usual one).

Theorem 3.4. *An extended pseudo-quasisemi metric space (X, d) is $\mathbf{T}_1$ if and only if for each pair of distinct points x and y in X, $d(x, y) > 0$.*

Proof. If $d(x, y) > 0$ for all $x \neq y$, then y does not belong to the ball $S(x, d(x, y))$. Since also $d(y, x) > 0$, we get x does not belong to the ball $S(y, d(y, x))$. Hence, a topological space (X, τ_d) is T_1.

If a topological space (X, τ_d) is $\mathbf{T}_1$, then it follows easily that $d(x, y) > 0$ for all $x \neq y$ in X.

□

Remark 3.1. (1) Let (X, τ) be a topological space and $p \in X$. By Theorem 3.1, a topological space (X, τ) is T_1 at p if and only if (X, τ) is T_1.

(2) Let (X, d) be an extended pseudo-quasi-semi metric space.

(i) By Theorem 3.3 and Theorem 3.4, (X, d) is T_1 (in our sense) implies (X, d) is $\mathbf{T}_1$ (in the usual sense) but the reverse of implication is not true. Let $X = \{x, y\}$ with $x \neq y$ and $d(x, y) = 1, d(y, x) = 3, d(x, x) = 0 = d(y, y)$. Then, by Theorem 3.3 and Theorem 3.4, (X, d) is $\mathbf{T}_1$ (in the usual sense) but (X, d) is not T_1 (in our sense).

4. Conclusions

In this paper, we gave characterization of each of local T_1 extended pseudo-quasi-semi metric space, T_1 extended pseudo-quasi-semi metric space, and the usual $\mathbf{T}_1$ extended pseudo-quasi-semi metric space. Moreover, by Theorem 3.3, 3.4, and Remark 3.1, T_1 extended pseudo-quasi-semi metric space implies $\mathbf{T}_1$ extended pseudo-quasi-semi metric space but the converse implication is not true, in general.

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